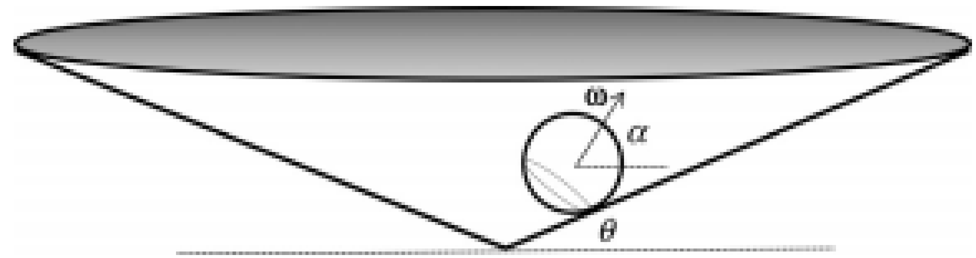


Balls Rolling in Cones

New-ish examples of learning-by-contrast

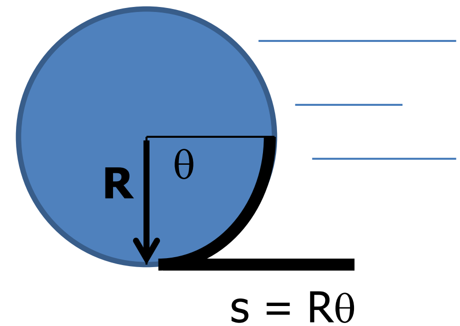
Gary D. White
American Association of Physics Teachers and
The George Washington University
gwhite@aapt.org, gwhite@gwu.edu

Physics of rolling

(rolling with friction, but no slipping)

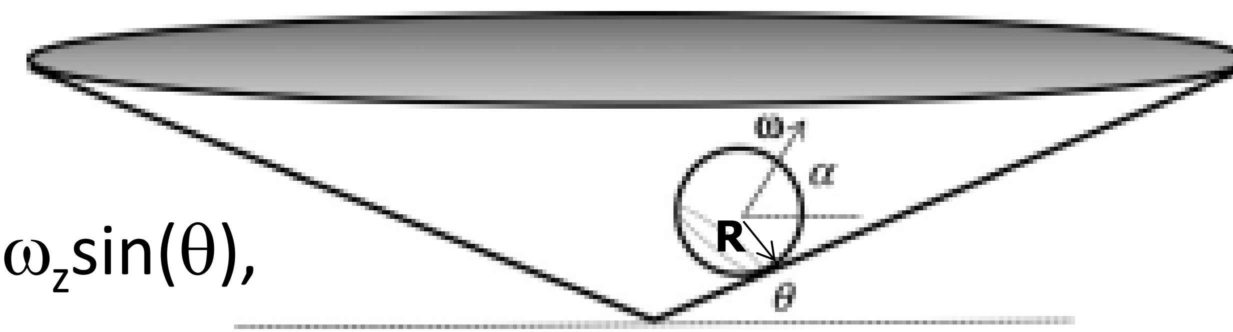
- Rolling without slipping, scalar

  $V_{cm} = R^*(d\theta/dt) = R^*\omega$



- Rolling without slipping, **vector**

  $\mathbf{V_{cm}} = \mathbf{R}\mathrm{x}\boldsymbol{\omega}$ **=** $[d\rho/dt, r d\phi/dt, dz/dt]$



$[R\omega_\phi \cos(\theta),$

$-R\omega_\rho \cos(\theta) - R\omega_z \sin(\theta),$

$R\omega_\phi \sin(\theta)]$

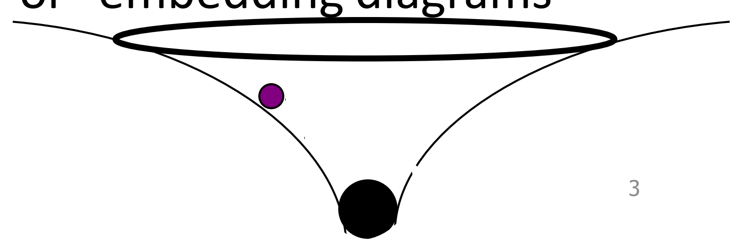
Rolling on cones and other funnels like the Spandex is good for demonstrating celestial phenomena:

- Orbits, precession
- Escape velocity
- Planetary Rings
- Roche Limit
- Density differentiation
- Early solar system agglomeration models
- Binary systems
- **Tidal Effects**
- For details see the Spandex trilogy:

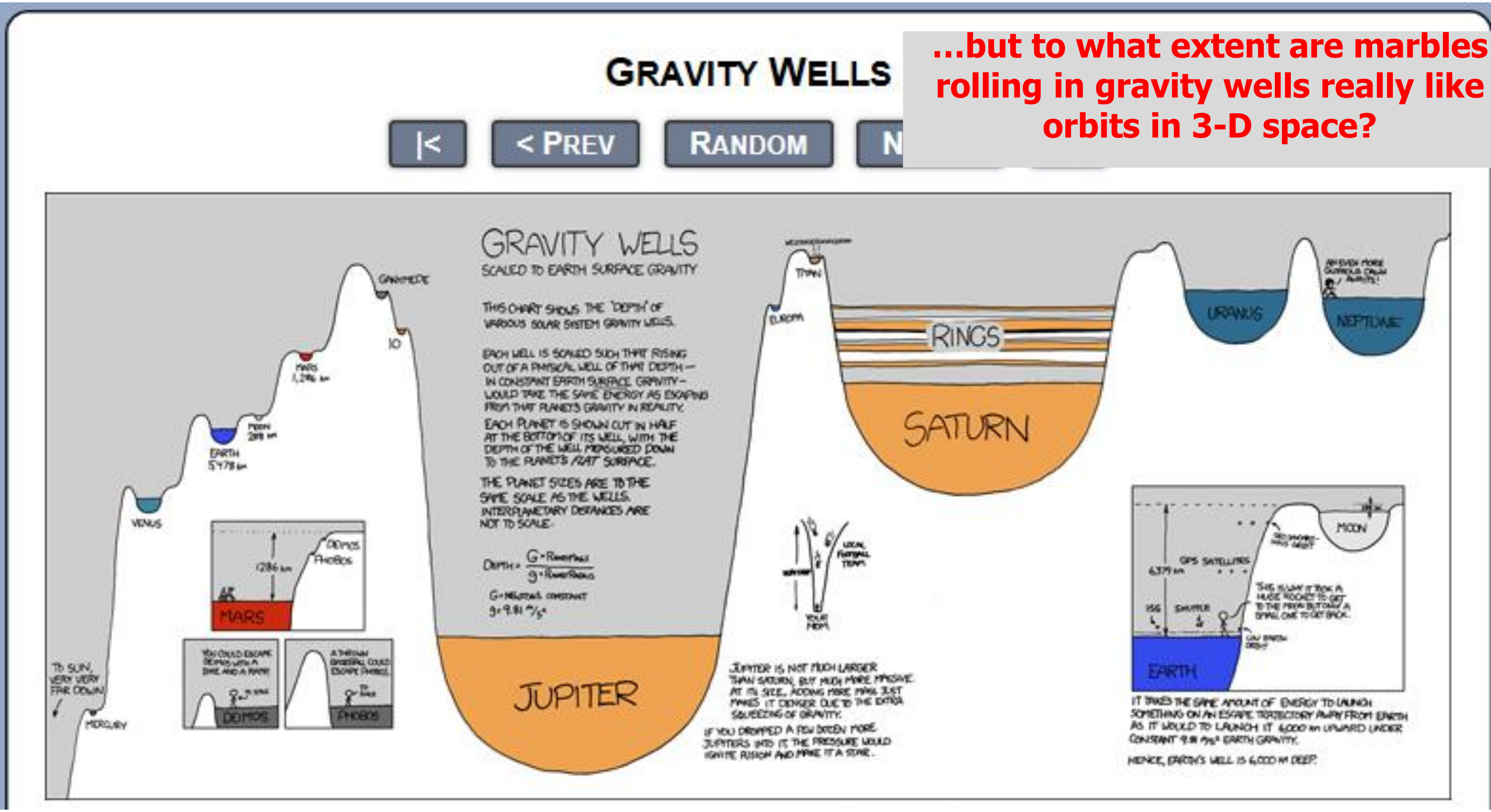
1) 'Modelling tidal effects' AJP **61**(4), '93
2) 'The shape of the Spandex and orbits upon its surface', AJP **70**(1), '02
3) 'On trajectories of rolling marbles in cones and other funnels' **81**(12), '13

by GDW and students

NOTE: "Gravity wells" rather than "curved space-time" or "embedding diagrams"



From XKCD (A webcomic of romance, sarcasm, math, and language, http://xkcd.com/681/)



...but to what extent are marbles rolling in gravity wells really like orbits in 3-D space?

Re-visiting Kepler's Law---discovery style

planets	period, T	radius from sun, R	T-squared	R-squared	T-cubed	R-cubed
	(in years)	(in earth-sun distances)				
Mercury	0.241	0.387	0.0580	0.150	0.0140	0.058
Venus	0.616	0.723	0.379	0.523	0.2338	0.378
Earth	1	1	1	1	1.0000	1.000
Mars	1.88	1.52	3.54	2.321	6.65	3.54
Jupiter	11.9	5.20	141.6	27.1	1685.16	140.8
Saturn	29.5	9.54	870.3	91.0	25672.38	867.9

So, in natural units, $T^2 = R^3$ for planets.

(In unnatural units, T^2 is merely proportional to R^3)

5

Kepler from Newton

- Of course, Newton's Laws gave us a fuller understanding of Kepler's finding, for circular orbits:

$\Sigma\mathbf{F}=m\mathbf{a}$

$\Rightarrow -GMm/R^2=-mV^2/R$

but $v = 2\pi R/T$

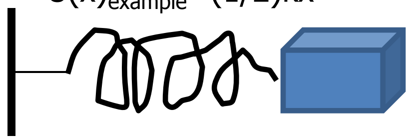
$\Rightarrow T^2$ is proportional to R^3

…and if the force law is different than inverse square, say if it is proportional to the reciprocal of the distance (like stretched spandex) or to the cube root of the distance (like unstretched spandex) or to the distance itself (like in a cone) then we get similar proportionality laws analogous to Kepler's laws that hold on that particular surface…even for rollers, not just frictionless sliders---why?

More about scalar rolling…

modelling one dimensional oscillations with scalar rolling without slipping

$U(x)_{example}=(1/2)kx^2$



- One-D motion

$$E_{1D} = \frac{1}{2}mV_x^{\;2} + U(x)$$

Diff. wrt time to get

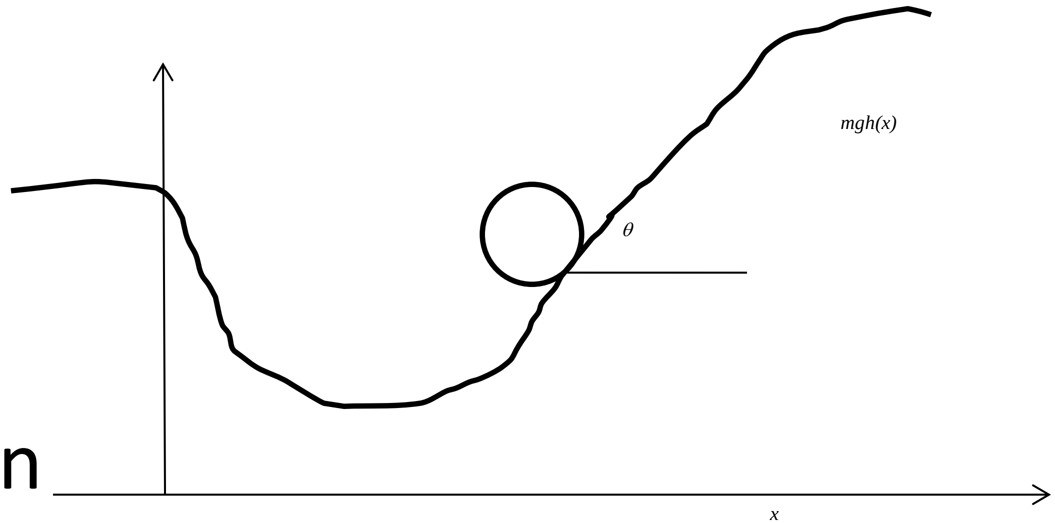
$$0 = \{m\ddot{x} + U'(x)\}V_x$$

Assume $x = x_0 + \delta$, then

$$0 = m\ddot{\delta} + U'(x_0) + \delta U''(x_0) + \ldots$$

So for small δ we get SHM with

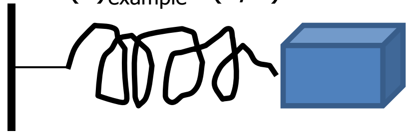
$$\Omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}$$



More about scalar rolling…

modelling one dimensional oscillations with scalar rolling without slipping

$U(x)_{example}=(1/2)kx^2$



- One-D motion

$$E_{1D} = \frac{1}{2}mV_x^{\;2} + U(x)$$

Diff. wrt time to get

$$0 = \{m\ddot{x} + U'(x)\}V_x$$

Assume $x = x_0 + \delta$, then

$$0 = m\ddot{\delta} + U'(x_0) + \delta U''(x_0) + \ldots$$

So for small δ we get SHM with

$$\Omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}$$

Rolling in a vertical plane in a valley given by h(x):

$$E_{roll} = \frac{1}{2}mV_x^{\;2} + \frac{1}{2}mV_y^{\;2} + \frac{1}{2}I\omega^2 + mgh(x)$$

but

$$V_y = V_x \tan(\theta) = V_x h'(x)$$

and no-slip rolling means

$$V^2 = V_x^{\;2} + V_y^{\;2} = (a\omega)^2$$

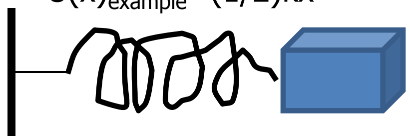
so $E_{roll} = \frac{1}{2}m\left(1+\frac{I}{ma^2}\right)\left(1+h'^2(x)\right)V_x^{\;2} + mgh(x)$

$$\Omega_{roll} = \sqrt{\frac{k_{roll}}{m_{roll}}} = \sqrt{\frac{mgh''(x_0)}{\left(m+I/a^2\right)\left(1+h'^2(x_0)\right)}}$$

More about scalar rolling…

modelling one dimensional oscillations with scalar rolling without slipping

$U(x)_{example}=(1/2)kx^2$



- One-D motion

$$E_{1D} = \frac{1}{2} m V_x^{\,2} + U(x)$$

So for small δ we get SHM with

$$\Omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}$$

Rolling in a vertical plane in a valley given by h(x):

so

$$E_{roll} = \frac{1}{2} m\left(1+\frac{I}{ma^2}\right)\left(1+h'^2(x)\right)V_x^{\,2} + mgh(x)$$

$$\Omega_{roll} = \sqrt{\frac{k_{roll}}{m_{roll}}} = \sqrt{\frac{mgh''(x_0)}{\left(m+I/a^2\right)\left(1+h'^2(x_0)\right)}}$$

Conclusion:

You can model motion of a mass at the end of a spring (1D motion) with a ball rolling in a vertical plane if

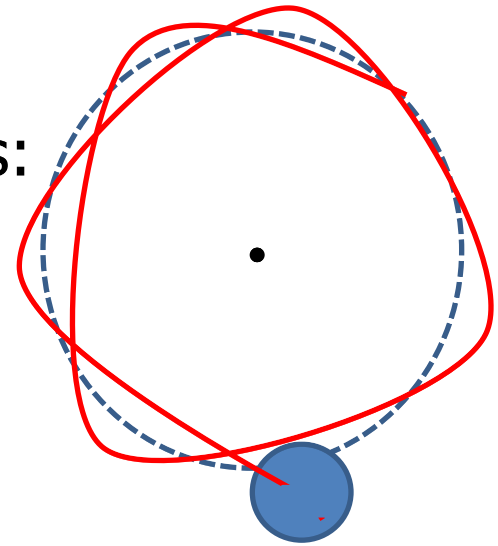
1) the shape of the hill matches the potential, and

2) if you "adjust" the mass and

3) if the derivatives of the hill are "small"

Now, vector rolling (that is, let's consider modelling planar motion in space with rolling motion on a cone or Spandex funnel…)

Write the energy as in the scalar case with some new orbital & spin terms:



$$E_{2D} = \frac{1}{2} m V_\rho^{\;2} + U(\rho) + L^2/(2m\rho^2) + spin\_terms$$

Diff. wrt time, assuming $\rho = R + \delta$

$$0 = m\ddot{\delta} + U'(R_0) + \boxed{U''(R_0)}\delta +$$
$$-L^2/(mR_0^{\;3}) + \boxed{3L^2/(mR_0^{\;4})}\delta + \ldots$$

$$\boxed{\Omega_{2D-oscillations} = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(R_0) + 3L^2/(mR_0^{\;4})}{m}}}$$

Again, SHM, constant terms give orbital frequency,

$$L^2 = mR_0^{\;3} U'(R_0) \to R_0 \dot{\phi}_{orbital}^2 = U'(R_0)/m \to T^2 \propto R_0 / U'(R_0)$$

If *U ~ 1/Ro, then we get Kepler's result: period square proportional to distance cubed*

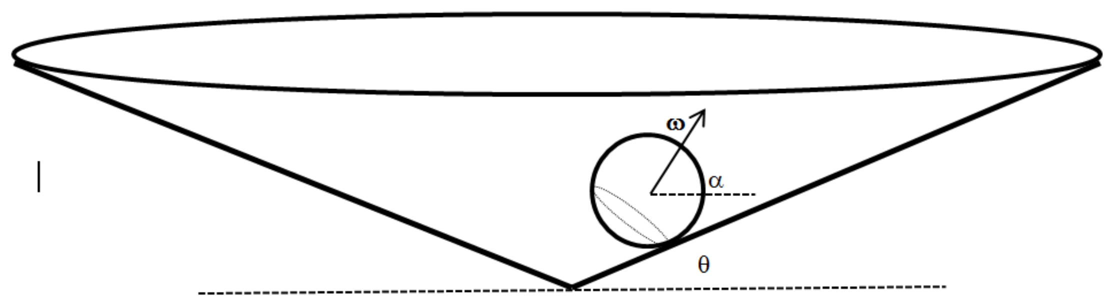
$\boxed{\text{coefficient of } \delta}$ gives frequency of small oscillations about orbit,

The details are a little complicated, but when rolling in a near-circular orbit in a cone we find

$$E_{rolling} = \frac{1}{2}(m + I/a^2)V_\rho^{\,2}(1+h'^2) + mgh(\rho) + \left(1 + I(1+h'^2)/(ma^2)\right)(L^2/(2m\rho^2) +$$

$$(I/a^2)(1+h'^2)(a\omega_z)\left(\frac{Lh'/m}{\rho\sqrt{1+h'^2}} + a\omega_z\right)$$

leading to,



$$R_0\dot{\phi}^2_{orbital} = gh'_0 \Big/ \left(1 + \frac{I}{ma^2}\frac{\cos(\alpha)}{\cos(\theta)\cos(\alpha-\theta)}\right)$$

Note the dependence on spin angle!

instead of Kepler's Law:

$$R_0\dot{\phi}^2_{orbital} = U'(R_0)/m$$

$$R_0\left(\frac{T}{2\pi}\right)^2 = GM/R_0^{\,2}$$

The details are a little complicated, but when rolling in a near-circular orbit in a cone we find

Conclusion:

You can model motion of a mass moving in a near circular celestial orbit with a ball rolling on a sheet of spandex or in a cone or other funnel if

1) the funnel shape, location, are carefully selected, and
2) if you "adjust" the mass and
3) if the derivatives of the hill are "small"
4) if you account for the spin of the marble

Note the dependence on spin angle!

instead of Kepler's Law:

$$R_0 \dot{\phi}^2_{orbital} = U'(R_0)/m$$

$$R_0 \left(\frac{T}{2\pi}\right)^2 = GM / R_0^{\,2}$$

The details are a little complicated, but when rolling in a near-circular orbit in a cone we find

Conclusion:

…f a mass moving in a near circular …g on a sheet of spandex or in …unnel if …fully selected, and

1) the funnel sh…

2) if you…

3) if the derivative…

4) if you account for the…

Better conclusion:

But that's beside the point…you can learn a lot of physics and have lots of fun besides by rolling marbles on spandex and exploring how its different from real gravity

$R_0 \ldots \left(1 + \ldots \frac{\ldots}{ma^2} \frac{\ldots\cos(\alpha)}{\cos(\theta)\cos(\alpha-\theta)} \right)$

Note the dependence on spin angle!

instead of Kepler's Law:

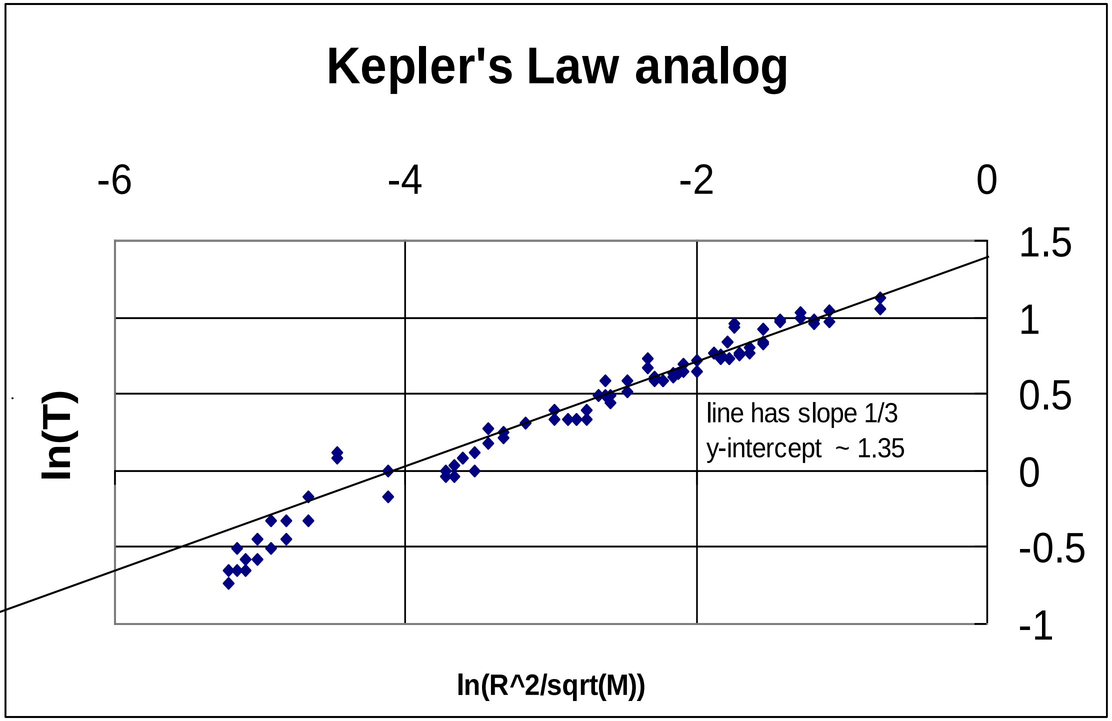
$R_0\left(\frac{T}{2\pi}\right)^2 = GM/R_0^2$

We determined Kepler's law analog for unstretched Spandex for circular orbits by doing some experiments…

- For fixed M, unstretched Spandex has $\ln(T)=(1/3)\ln(R^2)+b$
  - So, Spandex is $T^3/R^2 = k$ instead of $T^2/R^3 = c$.

notice how noisy the data is…

Experimenters can impart different spins to the marbles resulting in slightly different periods of orbit for the same radius…Let's try it on these cones…



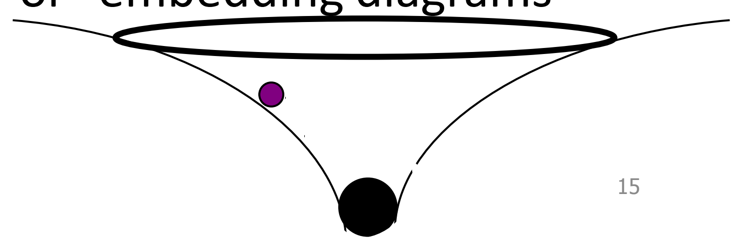
Rolling on cones and other funnels like the Spandex is good for demonstrating celestial phenomena:

- Orbits, precession
- Escape velocity
- Planetary Rings
- Roche Limit
- Density differentiation
- Early solar system agglomeration models
- Binary systems
- **Tidal Effects**
- For details see the Spandex trilogy:

1) 'Modelling tidal effects' AJP **61**(4), '93
2) 'The shape of the Spandex and orbits upon its surface', AJP **70**(1), '02
3) 'On trajectories of rolling marbles in cones and other funnels' **81**(12), '13

by GDW and students

NOTE: "Gravity wells" rather than "curved space-time" or "embedding diagrams"



Thanks to

- My students, especially Michael Walker, Tony Mondragon, Dorothy Coates, Darren Slaughter, Brad Boyd, Kristen Russell, Matt Creighton, Michael Williams, Chris Gresham, Randall Gauthier.
- Society of Physics Students (SPS) interns Melissa Hoffmann and Meredith Woy
- Aaron Schuetz, Susan White, Doug Singleton, Kendra Redmond, and Thomas Olsen
- SPS staff, AIP, AAPT, NSF, NASA, Fresno State Physics Department and
- You!